Counting Techniques: Poker Practice

September 25, 2023

To solidify your practice for counting problems, the following will work out the probabilities of receiving every distinct poker hand.

For each hand, specified in a table below, the goal is to compute (i) the number of ways of achieving the given hand, and (ii) the probability that each hand is drawn. Note, each of these are based on 5 card poker hands, so a hand is comprised of a random selection of 5 of the 52 total cards. (A deck of cards is divided into 4 suits, and each suit has 13 values, the numbers 2 through 10, and then J, Q, K, and A).

Hand	**Description**	**Example**
Royal Flush	The cards 10, J, Q, K, A of a single suite	10♣ J♣ Q♣ K♣ A♣
Straight Flush	Five cards in sequence of a single suite	3♥ 4♥ 5♥ 6♥ 7♥ A♠
Four of a Kind	All four suits of any particular value	2♥ 2♣ 2♠ 2♦ A♠
Full House	Two of one denomination, three of another	7♥ 7♣ 3♥ 3♦ 3♠
Flush	Five cards of the same suit	2♣ 3♣ 7♣ 10♣ A♣
Straight	Five consecutive cards of any suit	6♥ 7♣ 8♠ 9♥ 10♣
Three of a Kind	Three cards of the same denomination	5♥ 5♣ 5♠ 6♦ A♠
Two Pair	Two of one denomination, two of another	A♣ A♠ K♦ K♣ 8♠
Pair	Two cards of the same denomination	A♣ A♠ K♦ Q♣ 8♠
High Card	None of the previous hands	A♣ K♠ Q♦ 2♣ 8♠

If you complete all the hands, consider how many hands there are of each type if you exclude versions of that hand that correspond to an alternative, better hand. For instance, exclude straights that are also straight flushes or royal flushes, exclude pairs that are also full houses, etc.

Solution

Note that for all of the hands above, we are considering there being 5 card hands. As a result, we are selecting 5 cards from the 52 cards, without any concern for the order, and so there are a total of

$$N = \binom{52}{5} = 2598960,$$

possible 5 card hands. This denominator will remain unchanged throughout.

Royal Flush

Note that the denominations of the cards are fixed once a suit has been declared. That is, if we know that we have spades, then we need 10♠ J♠ Q♠ K♠ A♠. As a result, we know that there are as many royal flushes as there are suits (4). Put into a more formal framework, we can say that

$$N_A = \binom{4}{1} = \frac{4!}{1!3!} = 4.$$

So as a result,

$$P(A) = \frac{4}{2598960} = \frac{1}{649740}.$$

Straight Flush

Note that to form a straight flush, we first have to fix a suit. There are $\binom{4}{1} = 4$ total ways of doing this. Next, we need to pick which starting value we will use. Once a card has been selected as a starting value, the remaining cards are fixed. The start value ranges from A through to 10. Correspondingly, we have

$$N_A = \binom{4}{1}\binom{10}{1} = 4 \times 10 = 40.$$

As a result, we get that

$$P(A) = \frac{40}{2598960} = \frac{1}{64974}.$$

Uniqueness: Note that if we wish to exclude the possibility that a straight flush is also a royal flush, then we would take $N_A = 40 - 4 = 36$.

Four of a Kind

To form a hand with four of a kind, we need to pick all four cards of the denomination once that has been selected. Once that is complete, there are 48 more cards and we need to pick one of those at random to fill out the hand. Consequently, we get

$$N_A = \binom{13}{1} \times \binom{48}{1} = 13 \times 48 = 624.$$

As a result, we get that

$$P(A) = \frac{624}{2598960} = \frac{1}{4165}.$$

Full House

For a full house we need to have three of one denomination and two of the other. As a result, we need two of the 13 denominations represented. However, note that the ordering of these two does matter: one of them (say the first) will have to have 3 cards and the other (the second) will have to have 2 cards. Instead of $\binom{13}{2}$ then we are considering $P_{2,13}$. Then, for the first card, we will have $\binom{4}{3}$ ways of forming 3 of a kind, and for the second we will have $\binom{4}{2}$ ways of forming 2 of a kind. Taken all together this gives

$$N_A = P_{2,13} \times \binom{4}{3} \times \binom{4}{2} = 3744.$$

Then, to see the probability, we get

$$P(A) = \frac{3744}{2598960} = \frac{6}{4165}.$$

Flush

A flush necessitates drawing **any** five cards from the same suit. If we had a suit fixed, there would be $\binom{13}{5}$ ways of doing this, since we do not care about ordering. If we think about first choosing the suit, we have 4 ways of doing that, resulting in

$$N_A = 4 \times \binom{13}{5} = 5148.$$

As a result we would get

$$P(A) = \frac{5148}{2598960} = \frac{33}{16660}.$$

Uniqueness: Note that a flush can also be a straight flush (and as a result, a royal flush too). Including royal flushes, there are 40 total straight flushes, and so if we were to remove these from consideration we would get $N_A = 5148 - 40 = 5108$.

Straight

A straight necessitates drawing five cards in order, with each of any suit. Just as with the straight flush, there are 10 possible starting values for the straight. Once we have selected the starting value, then for each of the five cards we can pick any of the four suits, resulting in 4 choices each. That gives

$$N_A = 10 \times 4 \times 4 \times 4 \times 4 \times 4 = 10240.$$

This gives

$$P(A) = \frac{10240}{2598960} = \frac{128}{32487}.$$

Uniqueness: If a straight also happens to be a flush, then we have a straight flush. To consider only those which are not flushes, we would take $N_A = 10240 - 40 = 10200$.

Three of a Kind

To have three of a kind, we require to select a denomination, then any 3 of the four cards. The remaining two cards can be selected from any of the remaining 49 cards. However, note that if we were to take $\binom{13}{1} \times \binom{4}{3} \times \binom{49}{2}$ we would actually be doubling counting some hands. The issue is that whatever card is not selected in the $\binom{4}{3}$ is then counted in the $\binom{49}{2}$, doubling counting some hands.
Instead, we need to be somewhat more deliberate with our choices. Instead of selecting from the 4 and then 49, we can select from the 4 and then the 48 others. We will either select 3 or 4 cards from the 4, and correspondingly, 2 or 1 card from the 48. This gives

$$N_A = \binom{13}{1} \times \left(\binom{4}{3} \times \binom{48}{2} + \binom{4}{4} \times \binom{48}{1} \right) = 59,280.$$

This gives that

$$P(A) = \frac{59280}{2598960} = \frac{19}{833}.$$

Uniqueness: Note that three of a kind could also wind up being 4 of a kind or a full house. As a result, we could take $N_A = 59280 - 624 - 3744 = 54,912$.
We could also construct this total from a counting argument, directly. To get this we select the three of a particular denomination as above. Then, for the remaining two cards we would need them to come from separate denominations, giving $\binom{12}{2}$ different possibilities for those represented. Each of the two cards has 4 options within the denomination, giving

$$N_A = 13 \times \binom{4}{3} \times \binom{12}{2} \times \binom{4}{1}\binom{4}{1} = 54912$$

Two Pair

To have two pair, we require two separate denominations to be selected. The remaining card can be any of the remaining cards not previously selected. This gives

$$N_A = \binom{13}{2}\binom{4}{2} \times \binom{4}{2} \times \binom{44}{1} = 123,552.$$

This gives

$$P(A) = \frac{123,552}{2598960} = \frac{198}{4165}.$$

Note, we have constructed the two pair assuming that it does not construct a full house, directly with this technique. If you wish to include all possible full houses as well, these could be added back. Alternatively, this could be constructed as follows:

$$N_A = \binom{13}{2} \times \left(\binom{4}{3}\binom{4}{2}\binom{48}{0} + \binom{4}{2}\binom{4}{3}\binom{44}{0} + \binom{4}{2}\binom{4}{2}\binom{44}{1} \right) = 127,296.$$

Pair

To have a single pair, we must have one of the 13 denominations selected, and then any other 3 cards. Just as for the two pair, and the three of a kind, it is actually easier to construct exactly pairs (ignoring 3 and 4 of a kind). The reason being that once we have the denomination, we can select any 2 of the 4 possible cards, and then we need exactly 3 of the remaining 12 denominations represented. That is, we would get

$$N_A = \binom{13}{1} \times \binom{4}{2} \times \binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 1,098,240.$$

This gives

$$P(A) = \frac{1098240}{2598960} = \frac{352}{833}.$$

Now, if we wanted to include all pairs (including those which were two pair, three of a kind, four of a kind, and full houses) we could add these back. Direct construction is challenging here.

High Card

The easiest way to construct high cards is to simply take the total number of hands and subtract off all of the other possible hands, leaving just what is left. We will instead consider a direct construction.

Note that for denominations, we need to ensure that 5 separate values are represented. This will include 10 too many sets of denominations, since there are 10 straights represented there. For suit patterns, we can consider any suit pattern except for those that would define a flush; there are 4 of those, and so we get that there are $4^5 - 4 = 1020$ total suit patterns. Taking this together we get

$$N_A = \left(\binom{13}{5} - 10\right) \times 1020 = 1302540.$$

As a result, we have

$$P(A) = \frac{1302540}{2598960} = \frac{1277}{2548}.$$

Summary Solutions (Unique Counts)

Hand	**Count**
Royal Flush	4
Straight Flush	36
Four of a Kind	624
Full House	3744
Flush	5108
Straight	10,200
Three of a Kind	54,912
Two Pair	1,235,52
Pair	1,098,240
High Card	1,302,540
Total	2,598,960
Validation	$\binom{52}{5} = 2,598,960$